Discrete Mathematics（011122）

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- Set definition and its notation
- Subset and equality of sets
- Empty set and universal set
- Set operations (Intersection ∩, Union ∪, Relative complement −,
- Symmetric difference ⊕, Absolute complement ~)
- Venn diagram representation of set operations
- Basic identities of set operations
- Proof of partial identities

Set Definition

■**Cantor and Russell's discussions:**

- A set is one of the most fundamental concepts in mathematics, with no rigid definition. It is understood as a **collection of elements**, often denoted by A, B, etc.

■**Element:** The individual components of a set.

- $x \in A$ (x is an element of A)
- $x \notin A$ (x is not an element of A)

■**Finite Set:** A set with a limited number of elements.

■**Infinite Set:** A set with an unlimited number of elements.

■**Cardinality of Set** $|A|$: The number of elements in set A.

■**k-element Set:** A set containing k elements, where $k \geq 0$.

- **List Notation (enumeration method):**

  For example, A={a,b,c,d},N={0,1,2,…}

- **Set Builder Notation (descriptive method):**

  For example, {x | P(x)}, where x is a natural number.

- **Explanation:**

  (1)The elements in the set are **distinct**. For example, **{1,2,3}={1,1,2,3}**

  (2) The elements in the set **do not have an order**. For example

  {1,2,3}={3,1,2}={1,3,1,2,2}

  (3) The two methods should be selected based on the context.

- **Common Sets:**

  Natural numbers **N** (including 0), integers **Z**, positive integers Z^+ , rational numbers **Q**, irrational numbers **Q'**, real numbers **R**, all non-zero real numbers sets R^*, complex numbers **C**, intervals **[a,b],(a,b)** etc.

- **Subset (Inclusion):** $A \subseteq B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B)$
- **Not a Subset:** $A \not\subseteq B \Leftrightarrow \exists x\,(x \in A \wedge x \notin B)$
- **Equality of Sets:** $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$
- **Not Equal:** $A \neq B \Leftrightarrow A \not\subseteq B \vee B \not\subseteq A$
- **Proper Subset (Strict Subset):** $A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B$

Example: $A=\{1,2,3\}$, $B=\{\, x \mid x \in \mathrm{R} \wedge |x| \leq 1 \,\}$, $C=\{\, x \mid x \in \mathrm{R} \wedge x^2=1 \,\}$, $D=\{-1,1\}$, then we have $C \subseteq B$, $C \subset B$, $C \not\subseteq A$, $A \not\subseteq B$, $B \not\subseteq A$, $C = D$

- **Properties:**

  (1) $A \subseteq A$

  (2) $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

- **Empty set $\varnothing$**: A set that contains no elements.

  Example: $\{x \mid x^2<0 \wedge x \in R\}=\varnothing$

- **Theorem 1.1:** The empty set is a subset of any set.

  Proof: By contradiction.

  Suppose the contrary, i.e., there exists a set ***A*, such that $\varnothing \not\subseteq A$**, This means there exists an element ***x*, $x \in \varnothing$ and $x \notin A$, which is a contradiction.**

- **Corollary:** The empty set is unique.

  Proof: Suppose there exist two empty sets $\varnothing_1$ $\varnothing_2$, then $\varnothing_1 \subseteq \varnothing_2$ and $\varnothing_2 \subseteq \varnothing_1$, Hence $\varnothing_1=\varnothing_2$

- **Universal set E:** A set *E is called the universal set if all sets discussed in a problem is subset of E* , then $\forall A$ ($A \subseteq E$), then *E* is referred to as the universal set.

- **(Power Set)** It refers to the set of all subsets of a given set, including the empty set and the set itself， **express as** $P(A)$ **or** 2^A
- **Definition** $P(A) = \{ x \mid x \subseteq A \}$

Example:

$P(\varnothing) = \{\varnothing\}$,

$P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

$P(\{1,\{2,3\}\})=\{\varnothing,\{1\},\{\{2,3\}\},\{1,\{2,3\}\}\}$

- **Count**

If $|A| = n$, **then** $|P(A)| = 2^n$

- **Union:** $A \cup B = \{ x \mid x \in A \vee x \in B \}$
- **Intersection:** $A \cap B = \{ x \mid x \in A \wedge x \in B \}$
- **Relative Complement:** $A - B = \{ x \mid x \in A \wedge x \notin B \}$
- **Symmetric Difference:**

$$A \oplus B = (A-B) \cup (B-A) = (A \cup B) -(A \cap B)$$

- **Absolute Complement:** $\sim A = E - A = \{ x \mid x \notin A \}$

e.g. **Example1：**

Design E={0,1, … ,9}, A={0,1,2,3}, B={1,3,5,7,9},

Then： $A \cup B$ ={0,1,2,3,5,7,9}, $A \cap B$ ={1,3}, $A - B$ ={0,2},

$A \oplus B$ ={0,2,5,7,9}, $\sim A$ ={4,5,6,7,8,9}, $\sim B$ ={0,2,4,6,8}

Instructions:

(1)Use **only parentheses.**

(2)**Order of Operations**:

- Priority (1):Parentheses
- Priority (2): Complement ~\sim~ and power set
- Priority (3): Other operations.
- Operations of the same priority are evaluated from **left to right**.

e.g. **Example 1: Let E={ x | x is a student at a university in Beijing}, A,B,C,D *are subsets from E*,**

A= { x | x is from Beijing}}, B= { x | x is a external student},

C= { x | x is a student in the Department of Mathematics

D= { x | x likes listening to music}.

- **Describe the characteristics of the following sets of students:**

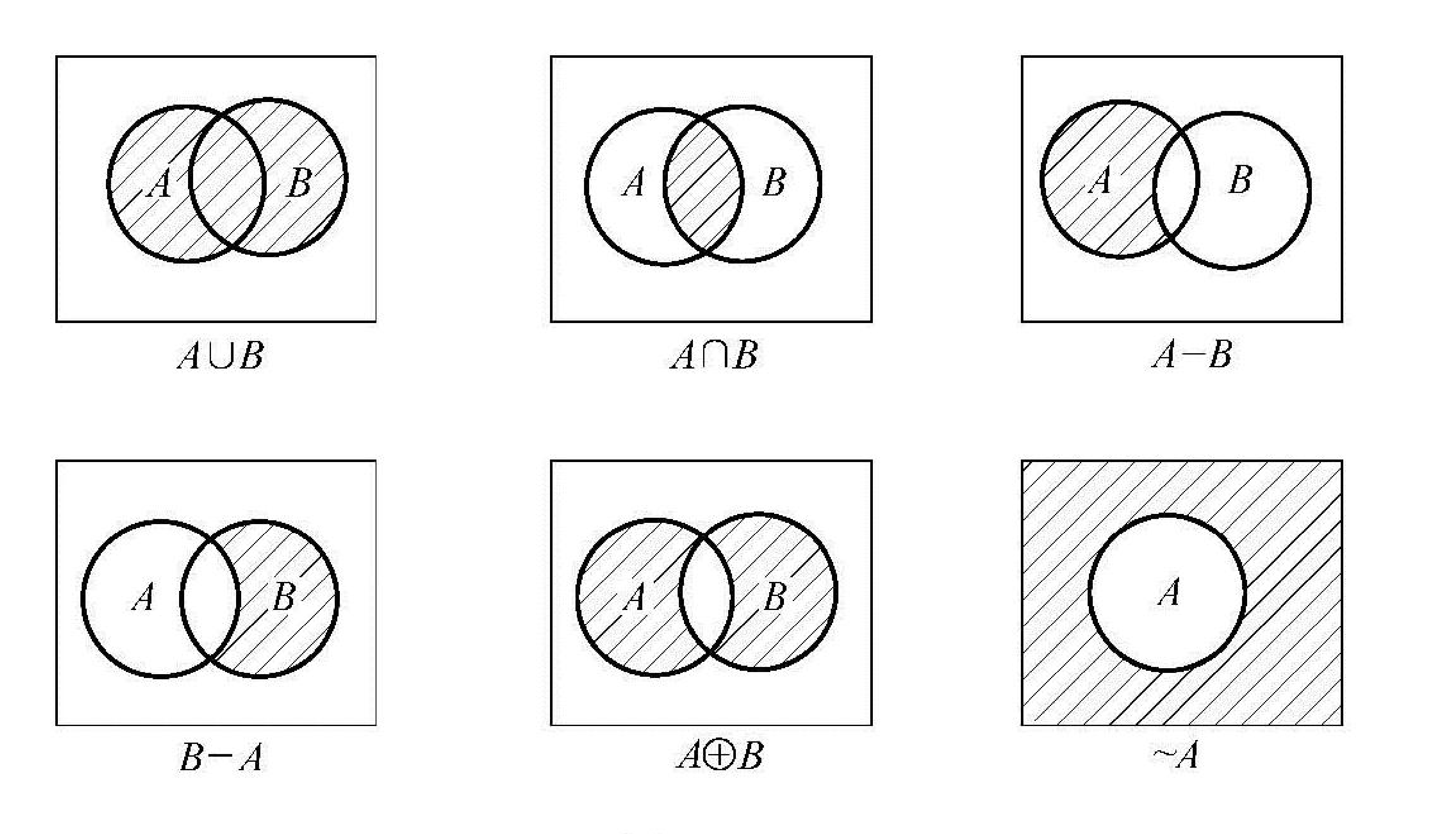
$(A \cup D) \cap \sim C$= { x | x is from Beijing or likes listening to music, but is not a mathematics student}

$\sim A \cap B$= { x | x is a non-local commuting student}

$(A - B) \cap D$= { x | x is a boarding student in Beijing and likes listening to music}

$\sim D \cap \sim B$= { x | x is a boarding student who does not like listening to music}

↳ Set Operations • Venn Diagram Representation



- Union and intersection operations can be extended to a **finite** **number** of sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x | x \in A_1 \vee x \in A_2 \vee \cdots \vee x \in A_n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x | x \in A_1 \wedge x \in A_2 \wedge \cdots \wedge x \in A_n\}$$

- Union and intersection operations can also be extended to countably **infinite** sets.

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = \{x | \exists i (i = 1, 2, \cdots) \, x \in A_i\}$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots = \{x | \forall i (i = 1, 2, \cdots) \, x \in A_i\}$$

e.g. **Example2:** Let $A_i=[0, 1/i)$, $B_i=(0, i)$, $i=1,2, \ldots$, then

$$\bigcup_{i=1}^{n} A_i = [0, 1) \qquad \bigcup_{i=1}^{\infty} A_i = [0, 1)$$

$$\bigcap_{i=1}^{n} A_i = [0, 1/n) \qquad \bigcap_{i=1}^{\infty} A_i = \{0\}$$

$$\bigcup_{i=1}^{n} B_i = (0, n) \qquad \bigcup_{i=1}^{\infty} B_i = (0, +\infty)$$

$$\bigcap_{i=1}^{n} B_i = (0, 1) \qquad \bigcap_{i=1}^{\infty} B_i = (0, 1)$$

1. Idempotent Law: $A \cup A=A$, $A \cap A=A$
2. Associative Law: $(A \cup B) \cup C=A \cup(B \cup C)$

   $(A\cap B) \cap C=A \cap(B \cap C)$
3. Commutative Law: $A \cup B=B \cup A$, $A \cap B=B \cap A$
4. Distributive Law: $A \cup(B \cap C)=(A \cup B) \cap (A \cup C)$

   $A \cap (B \cup C)=(A \cap B) \cup(A \cap C)$
5. Identity Law : $A \cup \varnothing=A$, $A \cap E=A$
6. Zero Law : $A \cup E=E$, $A \cap \varnothing=\varnothing$, **E is universal set**
7. Law of Excluded Middle: $A \cup {\sim}A=E$

8. Law of Contradiction: $A \cap \sim A = \varnothing$

9. Absorption Law: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

10. De Morgan's Laws:

Absolute Form: $\sim(B \cup C) = \sim B \cap \sim C$, $\sim(B \cap C) = \sim B \cup \sim C$

Relative Form: $A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

11. Complementation Law: $\sim\varnothing = E$, $\sim E = \varnothing$

12. Double Negation Law: $\sim\sim A = A$

13. Complement and Intersection Conversion Law: $A - B = A \cap \sim B$

14. Identities of Symmetric Difference

(1) Commutative Law: $A \oplus B = B \oplus A$

(2) Associative Law : $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(3) Distributive Law of Intersection over Symmetric Difference:

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

(4) $A \oplus \varnothing = A$, $A \oplus E = \sim A$

(5) $A \oplus A = \varnothing$, $A \oplus \sim A = E$

Note: The union operation ∪ does not distribute over the symmetric difference⊕, A counter example is given below:

$A=\{a,b,c\}$, $B=\{b,c,d\}$, $C=\{c,d,e\}$

$A \cup (B \oplus C) = \{a,b,c\} \cup \{b,e\} = \{a,b,c,e\}$

$(A \cup B) \oplus (A \cup C) = \{a,b,c,d\} \oplus \{a,b,c,d,e\} = \{e\}$, not equal

15. $A \subseteq A \cup B$, $B \subseteq A \cup B$.

16. $A \cap B \subseteq A$, $A \cap B \subseteq B$.

17. $A - B \subseteq A$.

18. $A \cup B = B \Leftrightarrow A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A - B = \varnothing$.

19. $A \oplus B = A \oplus C \Leftrightarrow B = C$.